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MODEL OF ELASTOPLASTIC DEFORMATION OF MATERIALS, BASED ON THE GAUGE THEORY OF DEFECTS WITH ALLOWANCE FOR ENERGY DISSIPATION

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Mathematical models of plasticity and creep for the case of small deformations are proposed on the basis of the gauge theory of defects with allowance for energy dissipation. It is assumed that plasticity is related to the motion of dislocations, which occurs without changes in volume. In the creep model, the motion of dislocations can proceed with changes in volume, and the "extra" volume is entrained (brought) by point defects. With the help of Godunov's generalized thermodynamic approach, it is shown that the proposed plasticity model is hyperbolic according to Friedrichs.

Key words: *gauge theory, elasticity, plasticity, creep, hyperbolicity.*

Introduction. Currently, there is elevated interest in constructing mathematical models of plasticity with allowance for the internal structure of the material. Three research directions can be identified in this field: geometric approach [1], gauge theory of defects [2], and generalized thermodynamic approach [3, 4]. The geometric approach was suggested by K. Kondo and B. A. Bilby and is based on a comparison of an elastic medium containing defects and a non-euclidean space with curvature and twisting. In [1], this approach was generalized to unsteady processes, and evolution equations for curvature, twisting, and nonmetricity tensors were obtained. The gauge theory of defects developed by A. Cadic and D. Edelen [3] allows one to describe an elastic medium with defects. Note, this theory is a Hamiltonian one, whereas plastic deformation leads to energy dissipation. Dissipative processes in the gauge theory of defects were taken into account in [5–7]. Godunov et al. [3, 4, 8] put forward a generalized thermodynamic approach to constructing inelastic models. The requirement of hyperbolicity of the system of equations and the law of energy conservation allow one to obtain closing relations in the form of differential equations for the field of defects.

Models of the elastoplastic behavior of materials, based on the gauge theory of defects with allowance for energy dissipation, are considered in the present paper.

1. Equations of the Gauge Theory of Defects. The Lagrangian of an isotropic elastic body in the case of small deformations $\partial u_j/\partial x_j \ll 1$ is determined by the formula [9]

$$L_e = \int dV\Big[\frac{1}{2}\,\rho\dot{u}_i\dot{u}_i - \frac{\lambda}{2}\,\frac{\partial u_i}{\partial x_i}\,\frac{\partial u_j}{\partial x_j} - \frac{\mu}{2}\Big(\frac{\partial u_i}{\partial x_j}\,\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}\,\frac{\partial u_j}{\partial x_i}\Big)\Big], \tag{1.1}$$

where u_i are the components of the displacement vector, $\dot{u}_i = \partial u_i/\partial t$, λ and μ are the Lamé coefficients, and ρ is the density. Summation in formula (1.1) and further formulas is performed over repeated indices.

Lagrangian (1.1) is invariant under translation by a constant vector $\boldsymbol{h} = h_i\boldsymbol{e}_i$ and under rotation by a constant vector $\boldsymbol{\Omega} = \Omega_i\boldsymbol{e}_i$ ($\boldsymbol{e}_i$ are the basis vectors of the Cartesian coordinate system). The components of the displacement vector under these transformations are determined by the relations

$$u_i' = u_i + h_i, \qquad u_i' = u_i + \varepsilon_{ijk}\Omega_j u_k, \tag{1.2}$$

where ε_{ijk} is an absolutely antisymmetric Levi-Civita tensor. In the case of local (gauge) transformations of translation $h_i = h_i(x_j, t)$ and rotation $\Omega_i = \Omega_i(x_j, t)$, invariance of Lagrangian (1.1) under transformations (1.2)

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is violated. To recover invariance, gauge fields are introduced, where the usual derivatives $\partial_j u_i$ are replaced by covariant derivatives $D_j u_i$. After substitution of $\partial_j u_i \to D_j u_i$ into the Lagrangian, its invariance is recovered.

The gauge fields related to translation and rotation determine the fields of dislocations and disclinations, respectively [2]. Experiments show that disclinations almost never arise in metals. The reason is that the elastic energy of disclinations is rather high [10] $E_\Omega \sim \mu H^2 \Omega^2$ [E_Ω is the elastic energy of a disclination per unit length, H is the reference size of the body, and Ω is the rotation angle (Frank vector)]; therefore, generation of disclinations in an ideal crystal is energetically inefficient. At the same time, the elastic energy per unit length of a dislocation is rather low $E_b \sim \mu b^2 \ln{(H/b)}$ (b is the Burgers vector, which has the order of interatomic distances a), and dislocations readily emerge and move in crystals. In addition to dislocations and disclinations, there are point defects (vacancies, inclusions) and pores whose elastic energy is small $E_a \sim \mu a^3$ (a is the interatomic distance for a vacancy or inclusion or the pore size). Thus, carriers of plastic (inelastic) deformation in metals are either dislocations or point defects and pores, and there are no disclinations. Based on this fact, only local translation transformations are considered below.

Following [2, 5–7], we construct a Lagrangian invariant under local translation transformations

$$u_i' = u_i + h_i(x_j, t) \tag{1.3}$$

by replacing partial derivatives in (1.1) by covariant derivatives

$$\frac{\partial u_i}{\partial t} \longrightarrow D_4 u_i = \frac{\partial u_i}{\partial t} + \beta_{4i}, \qquad \frac{\partial u_i}{\partial x_j} \longrightarrow D_j u_i = \frac{\partial u_i}{\partial x_j} + \beta_{ji}. \tag{1.4}$$

Here, we introduce gauge (compensating) fields $\beta_{4i}(x_k, t)$ and $\beta_{ji}(x_k, t)$, which can be related to the Lagrangian

$$L_d = \frac{1}{2} \int (B J_{ji} J_{ji} - C \alpha_{ij} \alpha_{ij})\, dV \tag{1.5}$$

(B and C are constants). The quantities

$$J_{ij} = -\Big(\frac{\partial \beta_{ij}}{\partial t} + \frac{\partial \beta_{4j}}{\partial x_i}\Big), \qquad \alpha_{ij} = \varepsilon_{ikl} \frac{\partial \beta_{lj}}{\partial x_k} \tag{1.6}$$

have the meaning of the flux and density of dislocations, respectively. Such an interpretation follows from the relations

$$\int\limits_S \alpha_{ij} n_i\, dS = \int\limits_S \varepsilon_{ikl} \frac{\partial \beta_{lj}}{\partial x_k} n_i\, dS = \oint \beta_{lj}\, dx_l = B_j,$$

$$-\int\limits_{\partial S} J_{ij}\, dx_i = \oint \frac{\partial \beta_{ij}}{\partial t}\, dx_i = \frac{\partial B_j}{\partial t},$$

where B_j is the total Burgers vector of dislocations crossing the area S bounded by the contour ∂S. Formulas for transformation of the gauge fields $\beta_{4i}(x_k, t)$ and $\beta_{ji}(x_k, t)$ follow from the condition of invariance of the gauge derivatives (1.4) under transformations Lamé)

$$\beta_{4i}' = \beta_{4i} - \frac{\partial h_i}{\partial t}, \qquad \beta_{ji}' = \beta_{ji} - \frac{\partial h_i}{\partial x_j}. \tag{1.7}$$

Substituting (1.7) into formulas (1.6), we can prove by direct calculations that J_{ij} and α_{ij} remain invariant under transformations Lamé). It follows from here that the Lagrangian

$$L = L_e(D_4 u_i,\, D_j u_i) + L_d(J_{ij}, \alpha_{ij}) \tag{1.8}$$

is invariant under transformations Lamé) and describes an elastic medium with dislocations, $L_e(D_4 u_i, D_j u_i)$ is obtained from Lagrangian (1.1) by means of substitution (1.4), and $L_d(J_{ij}, \alpha_{ij})$ is determined by formula (1.5).

The Euler–Lagrange equations [11]

$$\frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) + \frac{\partial}{\partial x_j}\Big(\frac{\partial L}{\partial q_{i,j}}\Big) - \Big(\frac{\partial L}{\partial q_i}\Big) = 0 \tag{1.9}$$

for the variables $q_i = \{u_i, \beta_{4i}, \beta_{ij}\}$ are found from the condition of extremality $\delta S = 0$ of the action $S = \displaystyle\int L\, dt$ and have the form

$$B\,\frac{\partial}{\partial x_j}\Big(\frac{\partial \beta_{ji}}{\partial t}+\frac{\partial \beta_{4i}}{\partial x_j}\Big)=\rho\Big(\frac{\partial u_i}{\partial t}+\beta_{4i}\Big),$$

$$B\,\frac{\partial}{\partial t}\Big(\frac{\partial \beta_{ji}}{\partial t}+\frac{\partial \beta_{4i}}{\partial x_j}\Big)=-C\varepsilon_{jkl}\,\frac{\partial \alpha_{li}}{\partial x_k}+\sigma_{ij}, \tag{1.10}$$

$$\rho\,\frac{\partial}{\partial t}\Big(\frac{\partial u_i}{\partial t}+\beta_{4i}\Big)=\frac{\partial \sigma_{ij}}{\partial x_j}.$$

The following formulas are valid for the stress tensor σ_{ij} entering into (1.10):

$$\sigma_{ij}=-p\delta_{ij}+S_{ij},\qquad p=-K\varepsilon^e_{kk},\qquad S_{ij}=2\mu e^e_{ij},\qquad K=\lambda+2\mu/3,$$
$$e^e_{ij}=\varepsilon^e_{ij}-\frac{1}{3}\,\varepsilon^e_{kk}\delta_{ij},\qquad \varepsilon^e_{ij}=\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big)-\varepsilon^p_{ij},\qquad \varepsilon^p_{ij}=\frac{1}{2}\,(\beta_{ij}+\beta_{ji}). \tag{1.11}$$

Here p is the pressure, S_{ij} is the stress-tensor deviator, and K is the volume elasticity modulus; the superscript e indicates elastic deformations. The zero value of surface integrals in the equation $\delta S=0$ allows us to determine the boundary conditions

$$f_i=\sigma_{ij}n_j,\qquad n_i\varepsilon_{kil}\varepsilon_{nml}\,\frac{\partial \beta_{mj}}{\partial x_n}=0. \tag{1.12}$$

System (1.10)–(1.12) coincides with the corresponding linearized system of equations obtained in [2] for the case of finite deformations. It should be noted that not all equations in (1.10) are independent. If we differentiate the first equation in (1.10) in time $\partial/\partial t$, differentiate the second equation with respect to the coordinate $\partial/\partial x_j$, and subtract one from the other, we obtain the third equation (more exactly, three equations, because the subscript i takes the values from 1 to 3). This result is a consequence of the second Nöther theorem (see [12]) according to which there are n identities among the Euler–Lagrange equations if the action S is invariant with respect to a group of transformations depending on n arbitrary functions. In the case considered, n Euler–Lagrange equations can be expressed via the remaining equations. In our case, the Lagrangian L and the action $S=\displaystyle\int L\,dt$ are invariant with respect to the group of transformations Lamé) depending on three arbitrary functions $h_i(x_j,t)$; therefore, there are three identities in (1.10). Thus, the number of independent equations in (1.10) is smaller than the number of independent variables, and they have to be supplemented by three equations, which are called gauge conditions. The Lorentz gauge equations or the Coulomb gauge [13] are used most frequently. In the present work, we use the Coulomb gauge $\beta_{4i}=0$. Then, it follows from the first equation in (1.7) that $\beta'_{4i}=\beta_{4i}=0$, if $\partial h_i/\partial t=0$, and the gauge transformation has the form

$$u'_i=u_i+h_i(x_j),$$

where h_i is independent of time. With allowance for the above-mentioned facts, Eqs. (1.10) transform to the following equations:

$$B\,\frac{\partial^2\beta_{ji}}{\partial t^2}=-C\varepsilon_{jkl}\,\frac{\partial \alpha_{li}}{\partial x_k}+\sigma_{ij},\qquad \rho\,\frac{\partial^2 u_i}{\partial t^2}=\frac{\partial \sigma_{ij}}{\partial x_j},\qquad \beta_{4i}=0.$$

Lagrangian (1.8) remains invariant under translation transformations; therefore, according to the first Nöther theorem (see [12, 14]), the following law of conservation of energy and momentum is valid:

$$\frac{\partial T^\beta_\alpha}{\partial x^\beta}=0,\qquad T^\beta_\alpha=\frac{\partial L}{\partial q_{i,\beta}}\,\frac{\partial q_i}{\partial x_\alpha}-L\delta^\beta_\alpha,\qquad \{\alpha,\beta\}=\{1,2,3,4\}. \tag{1.13}$$

Here, $T^4_4=E$ is the energy and T^k_4 is the momentum of the medium:

$$E=\frac{\partial L}{\partial \dot u_i}\Big(\frac{\partial u_i}{\partial t}\Big)+\frac{\partial L}{\partial \dot\beta_{ji}}\Big(\frac{\partial \beta_{ji}}{\partial t}\Big)-L=\frac{\rho}{2}\Big(\frac{\partial u_i}{\partial t}\Big)^2+\frac{B}{2}\Big(\frac{\partial \beta_{ij}}{\partial t}\Big)^2+E_1\Big(\varepsilon^e_{ij}\Big)+E_2(\alpha_{ij}),$$
$$T^k_4=\frac{\partial L}{\partial u_{i,k}}\Big(\frac{\partial u_i}{\partial t}\Big)+\frac{\partial L}{\partial \beta_{ji,k}}\Big(\frac{\partial \beta_{ji}}{\partial t}\Big)=-\sigma_{ik}\,\frac{\rho}{2}\Big(\frac{\partial u_i}{\partial t}\Big)-S_{jki}\Big(\frac{\partial \beta_{ij}}{\partial t}\Big). \tag{1.14}$$

We introduced the notation

$$S_{jki}=-C\varepsilon_{jkl}\alpha_{li},\qquad E_1(\varepsilon^e_{ij})=(\lambda/2)(\varepsilon^e_{kk})^2+\mu\varepsilon^e_{ij}\varepsilon^e_{ij},\qquad E_2(\alpha_{ij})=(C/2)\alpha_{ij}\alpha_{ij} \tag{1.15}$$

[S_{jki} is the tensor of the pair of forces generated by dislocations, $E_1(\varepsilon^e_{ij})$ is the elastic energy depending on elastic strains in the material, and $E_2(\alpha_{ij})$ is the elastic energy of interaction of dislocations]. Substituting (1.14) into (1.13), we obtain the following law of conservation of energy of an elastic medium with defects:

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial x_k}(\sigma_{ik} v_i + S_{jki}\dot\beta_{ji}),$$
$$E = \frac{\rho}{2}\, v_i^2 + \frac{B}{2}\,\dot\beta_{ij}^2 + E_1(\varepsilon^e_{ij}) + E_2(\alpha_{ij}), \qquad v_i = \frac{\partial u_i}{\partial t}, \qquad \dot\beta_{ij} = \frac{\partial \beta_{ij}}{\partial t}. \tag{1.16}$$

The total energy E is composed of the kinetic energy of motion of the medium and defects and also of the elastic energy of the medium and interacting defects (dislocations). The energy E is changed owing to the work of elastic stresses σ_{ij} on displacements $v_i\, dt$ and the moment of the pair of forces S_{jki} on plastic distortions $\dot\beta_{ji}\, dt$. The tensor S_{jki} can be expanded with respect to the indices i, j into symmetric and antisymmetric parts; therefore, the contribution to the work is made by the symmetric part $[d\varepsilon^p_{ij} = (\dot\beta_{ji} + \dot\beta_{ji})\, dt/2]$ and antisymmetric part $[d\omega^p_{ij} = (\dot\beta_{ji} - \dot\beta_{ji}) dt/2]$ of the distortion tensor.

2. Mathematical Model of Plasticity. The above-described model is a particular case of a more general gauge model of defects, containing dislocations and disclinations [2]. These models ignore energy dissipation in a plastic flow of a material. Owing to energy dissipation, there appears the Rayleigh dissipative force in the right side of the Euler–Lagrange equation:

$$\frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial \dot q_i}\Big) + \frac{\partial}{\partial x_j}\Big(\frac{\partial L}{\partial q_{i,j}}\Big) - \Big(\frac{\partial L}{\partial q_i}\Big) = -\frac{\partial D}{\partial \dot q_i} \tag{2.1}$$

[$D = D(\dot q_i)$ is a dissipative function]. In plastic deformation of metals, the dissipative function is usually approximated by a first-order homogeneous function of the plastic strain rate [16]:

$$D = Y_s\sqrt{(2/3)\dot\varepsilon^p_{ij}\dot\varepsilon^p_{ij}}, \qquad \dot\varepsilon^p_{ij} = \dot\beta_{(ij)} = (\dot\beta_{ji} + \dot\beta_{ji})/2.$$

Here $\dot\varepsilon^p_{ij} = \partial\varepsilon^p_{ij}/\partial t$ is the plastic strain rate and Y_s is the yield point. In this model, the independent variable is β_{ij}; we choose the dissipative function in a more general form

$$D = Y_s\sqrt{(2/3)\dot\beta_{ij}\dot\beta_{ij}}, \tag{2.2}$$

where $\dot\beta_{ji} = \dot\beta_{(ij)} + \dot\beta_{[ij]}$ and $\dot\beta_{[ij]} = (\dot\beta_{ij} - \dot\beta_{ji})/2$. In the particular case $\dot\beta_{[ij]} = 0$, expression (2.2) transforms to the dissipative function [16].

It is known from the experiment that plastic deformation occurs without changes in volume $\beta_{kk} = 0$; therefore, the Lagrangian L in (2.1) should be replaced by $\tilde L = L + \lambda_0\beta_{kk}$ (λ_0 is the Lagrange multiplier). Substituting the formula for $\tilde L$ into (2.1) and using the expressions for the Lagrangian L (1.5), (1.8) and D (2.2), we obtain the following equations that describe elastoplastic deformation [7]:

$$\rho\,\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j}, \qquad B\,\frac{\partial^2 \beta_{ji}}{\partial t^2} = S'_{ji} + S_{ji} - \sqrt{\frac{2}{3}}\, Y_s\,\frac{\dot\beta_{ji}}{\sqrt{\dot\beta_{ji}\dot\beta_{ji}}},$$
$$S'_{ji} = \frac{\partial}{\partial x_k}\Big(S_{jki} - \frac{1}{3}\, S_{lkl}\delta_{ji}\Big). \tag{2.3}$$

Here σ_{ij}, S_{ij}, and S_{jki} are determined in (1.11) and (1.15), and the boundary conditions are given by formulas (1.12). In a particular case, where the density of dislocations is independent of spatial coordinates $\partial\alpha_{ij}/\partial x_k = 0$ and the plastic strain rate is constant $\partial^2\beta_{ij}/\partial t^2 = 0$, the second equation of system (2.3) yields $\dot\beta_{[ij]} = 0$ and the Prandtl–Reuss relations [16] $\dot\varepsilon^p_{ij} = \sqrt{(3/2)\dot\varepsilon^p_{ij}\dot\varepsilon^p_{ij}}\,(S_{ij}/Y_s)$. Considering the second equation of system (2.3), we can find a simple mechanical analogy. If a load of mass m on a spring with rigidity k moves on a plane with a friction coefficient ν, its motion in the one-dimensional case is described by the equations

$$m\,\frac{\partial^2 x}{\partial t^2} = -kx - \frac{\nu m g \dot x}{|\dot x|},$$
$$\dot x = \ddot x = 0, \qquad -kx + f^r = 0, \qquad k|x| < \nu m g. \tag{2.4}$$

We can see that the second equation in (2.3) is similar to the first equation in (2.4); the sum of stresses $S'_{ji} + S_{ji}$ plays the role of the elastic force $-kx$, the quantity B plays the role of the mass m, and $\sqrt{2/3}\,Y_s$ plays the role of the sliding friction force νmg. If the load at a certain time is at rest $\dot{x} = 0$ and the elastic force acting on the load is smaller than the sliding friction force ($k|x| < \nu mg$), then the elastic force is balanced by the static friction force: $kx = f^r$, where $|f^r| < \nu mg$. The load remains at rest $\dot{x} = 0$ until the latter inequality is violated. For the complete analogy, system (2.3) should be supplemented by the equations

$$\frac{\partial \beta_{ij}}{\partial t} = 0, \qquad \frac{\partial^2 \beta_{ij}}{\partial t^2} = 0, \qquad \tilde{S}_{ij} - S^r_{ij} = 0 \qquad \text{for} \quad \tilde{S}_{ij}\tilde{S}_{ij} < \frac{2}{3}\,Y_s^2, \tag{2.5}$$

where $\tilde{S}_{ij} = S_{ij} + S'_{ij}$. The stress S^r_{ij} is similar to the static friction force f^r.

Energy dissipation in a plastic flow reduces the energy E, which is the sum of elastic and kinetic energies [see the second formula in (1.16)]. To obtain the law of energy conservation, we have to generalize this formula by including the thermal energy E_T:

$$E = \frac{\rho}{2}\,v_i^2 + \frac{B}{2}\,\dot{\beta}_{ij}^2 + E_1(\varepsilon^e_{ij}) + E_2(\alpha_{ij}) + E_T. \tag{2.6}$$

Heating of the medium generates the thermal pressure p_T; therefore, it is also necessary to change the second formula of system (1.11) as follows [17]:

$$p = p_x + p_T, \qquad p_x = -K\varepsilon^e_{kk}, \qquad p_T = \Gamma E_T, \qquad E_T = C_V \rho T. \tag{2.7}$$

Here Γ is the Grüneisen coefficient, C_V is the specific heat, and T is the temperature. Equations (2.6), (2.7) should be supplemented by the second law of thermodynamics and the Gibbs relation:

$$T\,\frac{dS}{dt} = Y_s\sqrt{\frac{2}{3}\,\dot{\beta}_{ij}\dot{\beta}_{ij}}, \qquad \frac{dE_T}{dt} = T\,\frac{dS}{dt} - p_T\,\frac{d}{dt}\Big(\frac{1}{\rho}\Big) \tag{2.8}$$

(S is the entropy density). Using (2.7), from the second formula in (2.8), we obtain the formula for entropy

$$S = S_0 + C_V\rho\ln\Big(\frac{T}{T_0}\Big(\frac{\rho_0}{\rho}\Big)^{\Gamma}\Big),$$

which allows us to express the thermal energy in formulas (2.6) and (2.7) as a function of density and entropy $E_T = E_T(\rho, S)$. Since we consider small deformations in this work, the total derivatives in formulas (2.8) can be replaced by partial derivatives $d/dt = \partial/\partial t + v_k\,\partial/\partial x_k \approx \partial/\partial t$. Then, the complete system of equations is written as

$$\begin{gathered}\rho\,\frac{\partial v_i}{\partial t} = \frac{\partial \sigma_{ij}}{\partial x_j}, \qquad B\,\frac{\partial \dot{\beta}_{ij}}{\partial t} = S_{ij} + \frac{\partial}{\partial x_k}\Big(S_{ikj} - \frac{1}{3}\,S_{lkl}\delta_{ij}\Big) - \sqrt{\frac{2}{3}}\,\frac{Y_s\dot{\beta}_{ij}}{\sqrt{\dot{\beta}_{ij}\dot{\beta}_{ij}}},\\ \frac{\partial \alpha_{lj}}{\partial t} = \varepsilon_{lki}\,\frac{\partial \dot{\beta}_{ij}}{\partial x_k}, \qquad \frac{\partial \varepsilon^e_{ij}}{\partial t} = \frac{1}{2}\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big) - \frac{\partial \varepsilon^p_{ij}}{\partial t}, \qquad \frac{\partial S}{\partial t} = \frac{Y_s}{T}\,\sqrt{\frac{2}{3}\,\dot{\beta}_{ij}\dot{\beta}_{ij}},\end{gathered} \tag{2.9}$$

where $\sigma_{ij} = -p\delta_{ij} + S_{ij}$, $S_{ij} = 2\mu e^e_{ij}$, $S_{ikj} = -C\varepsilon_{ikl}\alpha_{lj}$, and p is determined by formulas (2.7); the dot above a letter indicates a partial derivative in time. Multiplying the first equation of system (2.9) by v_i, the second equation by $\dot{\beta}_{ij}$, the third equation by $C\alpha_{lj}$, the fourth equation by σ_{ij}, and the fifth equation by T, and summing these equations, we obtain the law of energy conservation:

$$\frac{\partial E}{\partial t} = \frac{\partial \pi^k}{\partial x_k}, \qquad E = \frac{\rho v_i^2}{2} + \frac{B\dot{\beta}_{ij}^2}{2} + E_1 + E_2 + E_T, \qquad \pi^k = \sigma_{ik}v_i + S_{ikj}\dot{\beta}_{ij}. \tag{2.10}$$

Expression (2.10) is more general than (1.16), because it takes into account the change in the thermal energy of the medium E_T due to irreversible dissipation of energy in plastic deformation.

3. Mathematical Model of Creep. Creep is another example of the inelastic behavior where the metal flows like a liquid under the action of an applied stress. The creep process usually occurs at high temperatures. There are two classes of models for the creep description: phenomenological and microscopic. In phenomenological models, the governing relations are postulated on the basis of experimental data; in microscopic models, these relations are derived from the analysis of motion of defects in the field of external and internal stresses [10, 16]. Grinyaev and Chertova [18] used the gauge theory of defects for the creep description. It was assumed that creep is related to the motion of dislocations. In addition to dislocations, however, a significant effect on creep is exerted by point defects [10]. A mathematical model of creep, which takes into account the contributions of dislocations and point defects to creep, is proposed below on the basis of the gauge theory of defects.

First, we consider an isotropic elastic medium with dislocations, which is described by Lagrangian (1.5), (1.8). In contrast to plasticity, the creep process can occur under infinitesimal stresses and with changes in volume $\dot{\beta}_{kk} = \dot{\varepsilon}_{kk} \neq 0$; therefore, we choose the dissipative function in the form

$$D = \xi(\dot{\varepsilon}^p_{kk})^2 + \eta\dot{\beta}'_{ij}\dot{\beta}'_{ij}, \tag{3.1}$$

where $\dot{\beta}'_{ij} = \dot{\beta}_{ij} - (1/3)\dot{\beta}_{kk}\delta_{ij}$. In the particular case $\xi = \eta$, expression (3.1) coincides with the dissipative function used in [5, 18].

Substituting (1.5), (1.8), (3.1) into Eq. (2.1), we obtain the following equations for $q_i = \{u_i, \beta_{ij}\}$:

$$\rho\,\frac{\partial^2 u_i}{\partial t^2} = \frac{\partial \sigma_{ij}}{\partial x_j}, \qquad B\,\frac{\partial^2 \beta_{ij}}{\partial t^2} = \sigma'_{ij} + \sigma_{ij} - 2\eta\dot{\beta}'_{ij} - 2\xi\,\dot{\beta}_{kk}\delta_{ij},$$

$$\sigma_{ij} = -p\delta_{ij} + S_{ij}, \qquad p = -K\varepsilon^e_{kk}, \qquad S_{ij} = 2\mu e^e_{ij}, \qquad \sigma'_{ij} = C\Big(\frac{\partial^2 \beta_{ij}}{\partial x_k \partial x_k} - \frac{\partial^2 \beta_{kj}}{\partial x_k \partial x_i}\Big). \tag{3.2}$$

Assuming that $\beta_{ij} = \varepsilon^p_{ij} + \omega^p_{ij}$ in the second equation of system (3.2) and separating the spherical and deviatoric components, we rewrite this equation in the form

$$B\,\frac{\partial^2 e^p_{ij}}{\partial t^2} = S'_{(ij)} + S_{ij} - 2\eta\dot{e}^p_{ij}, \qquad B\,\frac{\partial^2 \varepsilon^p_{kk}}{\partial t^2} = -(p' + p + 2\xi\dot{\varepsilon}^p_{kk}),$$

$$B\,\frac{\partial^2 \omega^p_{ij}}{\partial t^2} = S'_{[ij]} - 2\eta\dot{\omega}^p_{ij}, \qquad S'_{ij} = \sigma'_{ij} + p'\delta_{ij}, \qquad S'_{ij} = S'_{(ij)} + S'_{[ij]}, \qquad p' = -\frac{1}{3}\,\sigma'_{kk}.$$

Inertial components in creep processes are small as compared to viscous components $B\ddot{e}^p_{ij} \ll \eta\dot{e}^p_{ij}$, $B\ddot{\varepsilon}^p_{kk} \ll \xi\dot{\varepsilon}^p_{ij}$, and $B\ddot{\omega}^p_{ij} \ll \eta\dot{\omega}^p_{ij}$; the creep equations take the form

$$\dot{e}^p_{ij} = \frac{S'_{(ij)} + S_{ij}}{2\eta}, \qquad \dot{\varepsilon}^p_{kk} = -\frac{p' + p}{2\xi}, \qquad \dot{\omega}^p_{ij} = \frac{S'_{[ij]}}{2\eta}. \tag{3.3}$$

The change in ε^p_{kk} in (3.3) is related to nonconservative motion of dislocations and should be accompanied by a flux of point defects (vacancies) to the dislocation, which entrain the "extra" ($\varepsilon^p_{kk} \neq 0$) material from the dislocation or bring it to the dislocation.

We obtain the equations for the motion of point defects. If introduction of a point defect changes the volume of the body by Ω_d and the concentration of defects is n_d, the strain in the material is determined by the formula

$$\dot{\varepsilon}_{kk} = \dot{\varepsilon}^e_{kk} + \dot{\varepsilon}^d_{kk}, \qquad \dot{\varepsilon}^d_{kk} = \Omega_d\dot{n}_d, \tag{3.4}$$

where ε^e_{kk} is the elastic strain in the lattice. If the total strain equals zero ($\dot{\varepsilon}_{kk} = 0$), we obtain $\dot{\varepsilon}^e_{kk} = -\dot{\varepsilon}^d_{kk}$ from (3.4). For interstitial atoms, we have $\Omega_d = \Omega_a > 0$ and $\varepsilon^e_{kk} < 0$ (the lattice is compressed); for vacancies, we have $\Omega_d = -\Omega_v$ and $\varepsilon^e_{kk} > 0$ (the lattice is extended). Compressing stresses $p = -K\varepsilon^e_{kk} > 0$ and extending stresses $p < 0$ act in the first and second case, respectively. Point defects are normally modeled by dilatancy centers [10]. In this case, the defect at the point x^0_i corresponds to the density of forces $f_i = -K\Omega_d\delta(x_i - x^0_i)$. If such a defect is in the field of elastic forces $p = -K\varepsilon^e_{kk}$, it corresponds to the elastic energy of interaction $E' = -K\Omega_d\varepsilon^e_{kk}$.

Since the concentration of defects n_d is usually high, defects can be described as a solid solution in an atomic lattice. Introducing the solution concentration $c = n_d/N \ll 1$ (N is the number of atoms in the lattice per unit volume), we write the chemical potential of the defect μ, as in the theory of weak solutions [10]:

$$\mu = T\ln c + \Omega_d p + \psi(T). \tag{3.5}$$

The conditions of equilibrium of point defects $\mu = \text{const}$ yields the formula for the equilibrium concentration

$$c = c_0(T)\exp(-p\Omega_d/T).$$

If $\nabla\mu \neq 0$, there arises a flux of point defects $\boldsymbol{j} = -T^{-1}n_d D\nabla\mu$, for which, with allowance for (3.5), we obtain

$$\boldsymbol{j} = -D\nabla n_d - T^{-1}n_d\Omega_d D\nabla p. \tag{3.6}$$

Using the continuity equation, we write the equation for the concentration of defects

$$\frac{\partial n_d}{\partial t} + \operatorname{div} \boldsymbol{j} = \dot{\theta}, \tag{3.7}$$

where $\boldsymbol{j}$ is determined by Eq. (3.6). The source term $\dot{\theta}$ is related to deposition or evaporation of atoms on the dislocation line $\dot{\theta} = \dot{\varepsilon}^p_{kk}/\Omega_d$, where $\dot{\varepsilon}^p_{kk}$ is determined by the second formula in (3.3). In the particular case $\boldsymbol{j} = 0$, Eqs. (3.4) and (3.7) yield $\dot{\varepsilon}^d_{kk} = \dot{\varepsilon}^p_{kk}$. As the boundary conditions for Eq. (3.7), we have to set $n_d|_\gamma$ or $\nabla n_d|_\gamma$ at the boundary γ. The emergence of defects to the surface is accompanied by the normal displacement of the surface with the velocity

$$v_n\Big|_\gamma = \Omega_d j\Big|_\gamma. \tag{3.8}$$

4. Symmetrization of Equations of Elastoplastic Deformation (2.9). The law of energy conservation (2.10) for system (2.9) allows us to use the generalized thermodynamic approach developed in [8, 19] to reduce this system to a form with symmetric matrices. The internal energy E (2.6) in the law of energy conservation (2.10) is a convex function of the eigenvariables $E = E(\rho v_i, B\dot{\beta}_{ij}, \varepsilon^e_{ij}, C\alpha_{ij}, S)$. Using the Legendre transformation

$$F^0 = w_i(\rho v_i) + b_{ij}(B\dot{\beta}_{ij}) + r_{ij}\varepsilon^e_{ij} + a_{ij}(C\alpha_{ij}) + TS - E, \tag{4.1}$$

we introduce the potential F^0 depending on the conjugate variables

$$F^0 = F^0(w_i, b_{ij}, r_{ij}, a_{ij}, T), \tag{4.2}$$

found by the formulas

$$w_i = \frac{\partial E}{\partial(\rho v_i)}, \qquad b_{ij} = \frac{\partial E}{\partial(B\dot{\beta}_{ij})}, \qquad r_{ij} = \frac{\partial E}{\partial \varepsilon^e_{ij}}, \qquad a_{ij} = \frac{\partial E}{\partial(C\alpha_{ij})}, \qquad T = \frac{\partial E}{\partial S}. \tag{4.3}$$

It follow from (4.1)–(4.3) that

$$dF^0 = \rho v_i\, dw_i + B\dot{\beta}_{ij}\, db_{ij} + \varepsilon^e_{ij}\, dr_{ij} + C\alpha_{ij}\, da_{ij} + S\, dT,$$

whence we find

$$\rho v_i = F^0_{w_i}, \qquad B\dot{\beta}_{ij} = F^0_{b_{ij}}, \qquad \varepsilon^e_{ij} = F^0_{r_{ij}}, \qquad C\alpha_{ij} = F^0_{a_{ij}}, \qquad S = F^0_T, \tag{4.4}$$

where the subscript of F^0 indicates the partial derivative, e.g., $F^0_{w_i} = \partial F^0/\partial w_i$, etc. In a similar manner, we consider three Legendre transformations constructed on the basis of three functions (2.10) $\pi^k = \sigma_{ik}v_i + S_{ikj}\dot{\beta}_{ij}$ and determine three potentials

$$\begin{gathered} F^k(w_i, b_{ij}, r_{ij}, a_{ij}) = w_i\sigma_{ik} + r_{ij}H_{ikj} + b_{ij}S_{ikj} + a_{ij}G_{ikj} - \pi^k, \\ w_i = \frac{\partial \pi^k}{\partial \sigma_{ik}}, \qquad r_{ij} = \frac{\partial \pi^k}{\partial H_{ikj}}, \qquad b_{ij} = \frac{\partial \pi^k}{\partial S_{ikj}}, \qquad a_{ij} = \frac{\partial \pi^k}{\partial G_{ikj}} \end{gathered} \tag{4.5}$$

(no summation over k is performed). Using the explicit form of the functions π^k (2.10) depending on the variables

$$\pi^k = \pi^k(\sigma_{ik}, H_{ikj}, S_{ikj}, G_{ikj}) \qquad [H_{ikj} = (\delta_{ik}v_j + \delta_{jk}v_i)/2, \quad G_{ikj} = C\varepsilon_{ikl}\dot{\beta}_{lj}], \tag{4.6}$$

we can show that the new variables in (4.5) coincide with the variables introduced previously by formulas (4.3). The quantities S_{ikj} are determined in (1.15), and the variable H_{ikj} was introduced in [8]. Using (4.5) and (4.6), we obtain

$$dF^k = \sigma_{ik}\, dw_i + S_{ikj}\, db_{ij} + H_{ikj}\, dr_{ij} + G_{ikj}\, da_{ij},$$

which yields the formulas

$$\sigma_{ik} = F^k_{w_i}, \qquad S_{ikj} = F^k_{b_{ij}}, \qquad H_{ikj} = F^k_{r_{ij}}, \qquad G_{ikj} = F^k_{a_{ij}}, \tag{4.7}$$

with the notation

$$F^k_{w_i} = \frac{\partial F^k}{\partial w_i}, \qquad F^k_{b_{ij}} = \frac{\partial F^k}{\partial b_{ij}}, \qquad F^k_{r_{ij}} = \frac{\partial F^k}{\partial r_{ij}}, \qquad F^k_{a_{ij}} = \frac{\partial F^k}{\partial a_{ij}}.$$

Substituting formulas (4.4) and (4.7) into the equations of elastoplastic deformation (2.9), we rewrite them in the new variables w_i, b_{ij}, r_{ij}, a_{ij}, and T:

$$\frac{\partial F^0_{w_i}}{\partial t} = \frac{\partial F^k_{w_i}}{\partial x_k}, \qquad \frac{\partial F^0_{b_{ij}}}{\partial t} = \frac{\partial F^k_{b_{ij}}}{\partial x_k} - \frac{1}{3}\frac{\partial F^k_{b_{ll}}}{\partial x_k}\delta_{ij} + r_{ij} - \frac{1}{3} r_{ll}\delta_{ij} - \sqrt{\frac{2}{3}}\, Y_s \frac{b_{ij}}{\sqrt{b_{ij}b_{ij}}},$$

$$\frac{\partial F^0_{a_{ij}}}{\partial t} = \frac{\partial F^k_{a_{ij}}}{\partial x_k}, \qquad \frac{\partial F^0_{r_{ij}}}{\partial t} = \frac{\partial F^k_{r_{ij}}}{\partial x_k} - \frac{1}{2}(b_{ij} + b_{ji}), \qquad \frac{\partial F^0_T}{\partial t} = Y_s\sqrt{\frac{2}{3}\, b_{ij}b_{ij}}\,. \tag{4.8}$$

Introducing the vector

$$q_i = (b_{11}, \ldots, b_{33}, a_{11}, \ldots, a_{33}, r_{11}, \ldots, r_{33}, w_1, w_2, w_3, T),$$

we can write Eqs. (4.8) as

$$\frac{\partial F^0_{q_i}}{\partial t} = \frac{\partial F^k_{q_i}}{\partial x_k} + \ldots,$$

where the dots stand instead of terms without derivatives, and the expression $-(1/3)(\partial F^k_{b_{ll}}/\partial x_k)\delta_{ij}$, which will be analyzed below. Following [8], we rewrite these equations as

$$F^0_{q_iq_j}\frac{\partial q_j}{\partial t} = F^k_{q_iq_j}\frac{\partial q_j}{\partial x_k} + \ldots. \tag{4.9}$$

The matrix $F^k_{q_iq_j}$ is symmetric, and matrix $F^0_{q_iq_j}$ is positively determined. The latter follows from the fact that the Legendre transformation (4.1) does not violate the function convexity. Therefore, if $E = E(y_i)$ is a convex function of the eigenvariables y_i, then $F^0 = F^0(q_i)$ is also a convex function of q_i. The symmetric form of system (4.8) is violated by the term $-(1/3)(\partial F^k_{b_{ll}}/\partial x_k)\delta_{ij}$ in the second equation of this system. Nevertheless, the symmetric form can be recovered if we add the following zero term to the right side of the third equation:

$$T^k_{ij} = -\frac{1}{3} F^k_{b_{ll}a_{ij}}\frac{\partial b_{mn}}{\partial x_k}\delta_{mn}.$$

To prove the identity $T^k_{ij} = 0$, we express the new variables in terms of the old variables with the help of (2.6) and (4.3):

$$w_i = v_i, \qquad b_{ij} = \dot\beta_{ij}, \qquad r_{ij} = \sigma_{ij}, \qquad a_{ij} = \alpha_{ij}. \tag{4.10}$$

From (4.10) and the equalities $\beta_{nn} = \dot\beta_{nn} = b_{nn} = b_{mn}\delta_{mn} = 0$, we obtain the sought identity

$$\frac{\partial b_{mn}}{\partial x_k}\delta_{mn} = -\frac{1}{3} F^k_{b_{ll}a_{ij}}\frac{\partial b_{mn}}{\partial x_k}\delta_{mn} = T^k_{ij} = 0, \tag{4.11}$$

where

$$\frac{1}{3} F^k_{b_{ll}a_{ij}} = \frac{1}{3}\frac{\partial}{\partial a_{ij}} S_{lkl} = \frac{C}{3}\varepsilon_{ijk}. \tag{4.12}$$

Similarly, from (4.7), (4.10), and (4.12) we obtain

$$-\frac{1}{3}\frac{\partial F^k_{b_{ll}}}{\partial x_k}\delta_{ij} = -\frac{1}{3}\frac{\partial S_{lkl}}{\partial x_k}\delta_{ij} = -\frac{C}{3}\varepsilon_{nmk}\frac{\partial a_{mn}}{\partial x_k}\delta_{ij}. \tag{4.13}$$

Taking into account equalities (4.11)–(4.13), we rewrite system (4.8) in a symmetric form:

$$\frac{\partial F^0_{w_i}}{\partial t} = \frac{\partial F^k_{w_i}}{\partial x_k}, \quad \frac{\partial F^0_{b_{ij}}}{\partial t} = \frac{\partial F^k_{b_{ij}}}{\partial x_k} - \frac{1}{3} F^k_{b_{ll}a_{mn}}\frac{\partial a_{mn}}{\partial x_k}\delta_{ij} + r_{ij} - \frac{1}{3} r_{ll}\delta_{ij} - \sqrt{\frac{2}{3}}\, Y_s\frac{b_{ij}}{\sqrt{b_{ij}b_{ij}}},$$

$$\frac{\partial F^0_{a_{ij}}}{\partial t} = \frac{\partial F^k_{a_{ij}}}{\partial x_k} - \frac{1}{3} F^k_{b_{ll}a_{ij}}\frac{\partial b_{mn}}{\partial x_k}\delta_{mn}, \qquad \frac{\partial F^0_{r_{ij}}}{\partial t} = \frac{\partial F^k_{r_{ij}}}{\partial x_k} - \frac{1}{2}(b_{ij} + b_{ji}), \tag{4.14}$$

$$\frac{\partial F^0_T}{\partial t} = Y_s\sqrt{\frac{2}{3}\, b_{ij}b_{ij}}.$$

Here, $F^k_{b_{ll}a_{ij}} = (1/3)C\varepsilon_{ijk}$. Since system (4.14) is symmetric and the function $F^0(q_i)$ is convex, it is hyperbolic according to Friedrichs (see [8]).

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